PARTIAL EIGENVALUES PROBLEM IN THE ANALYSIS OF HYDRODYNAMIC STABILITY DURING NATURAL CONVECTION

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A numerical method is outlined by which the first few eigenvalues in the spectrum of finite-difference approximation equations can be determined for the case of small perturbations in problems concerning natural convection in a homogeneous incompressible fluid.

In analyzing the hydrodynamic stability during heat convection, when the flow mode (monotonic or oscillatory instability) is to be established and when the critical value of the Rayleigh number is to be determined together with the corresponding wave number, it suffices to find the first few eigenvalues of the problem in the parameter λ, if the solution is assumed to be an exponential function of time $\exp(\lambda\tau)$. The Bubnov–Galerkin method, which is commonly used for this purpose, runs into certain difficulties in a number of cases (for example, in the problem of natural convection in a semiinfinite medium with a uniform injection).

In this article the author proposes a numerical method of solving such problems. The first few eigenvalues of a problem are determined by the fully stabilizing step-by-step power method [4]. The stabilization method in [5] is used for finding the eigenvectors of the problem directly from the finite-differences approximations to the system of transient equations for small perturbations.

We will consider the general case of natural convection and a transverse flow in a homogeneous liquid. In order to evaluate the effectiveness of this method, it was applied to several problems whose solutions had already been obtained earlier by other methods [1-3].

1. We consider a horizontally infinite plane layer of an incompressible homogeneous fluid. The vertical Z-axis will originate at the lower boundary of the liquid layer. A uniform injection (or ejection) at a velocity w_0 will be assumed to occur through the unequally heated boundary surfaces of this layer. If the liquid layer has no upper boundary, then we stipulate a temperature of the liquid at infinity.

The system of equations for the amplitudes of small temperature $\Theta(\tau, z)$ perturbations and velocity $\omega(\tau, z)$ (Z-component) perturbations, assuming the solution to be periodic in the horizontal plane, will be written as follows (with the wave number M):

$$\begin{gathered} -\frac{\partial}{\partial\tau} D\omega + D_1\omega - RM^2\Theta = 0, \\ \Pr\frac{\partial}{\partial\tau}\Theta - D_2\Theta + T_0'\omega = 0 \end{gathered} \tag{1}$$

for $\omega = \partial\omega/\partial z = 0$ at $z = 0, 1$ and with the corresponding boundary conditions for Θ.

The quantities in system (1) are dimensionless. As the characteristic units we choose H, ΔT, a/H, and H^2/ν for the length, the temperature, the velocity, and time, respectively.

The gradient $T_0'(z)$ of an unperturbed temperature distribution is:

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a) for the Rayleigh problem ($\alpha = 0$) with boundary conditions of the first kind [1] and of the third kind [2] for Θ,

$$T_0'(z) = -1, \tag{2}$$

b) for the generalized Rayleigh problem ($\alpha \neq 0$) with a uniform injection (or ejection) [3],

$$T_0'(z) = -\frac{\alpha}{1-\exp(-\alpha)} \exp(-\alpha z), \tag{3}$$

c) for a region without upper boundary and with ejection,

$$T_0'(z) = -\alpha \exp(-\alpha z). \tag{4}$$

We note that the problem cannot be formulated in terms of small perturbations as (1) for a layer without upper boundary and with either a stationary liquid or a uniform injection through the lower boundary surface, since under either of these conditions there is no steady distribution of unperturbed temperature.

System (1) can be conveniently rewritten in matrix form:

$$A\mathbf{X} - B\frac{\partial}{\partial\tau}\mathbf{X} = 0, \tag{5}$$

where A and B are matrices of second rank:

$$A = \begin{bmatrix} \frac{1}{K}D_1 & -K \\ KT_0' & -KD_2 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{K}D & 0 \\ 0 & -K\,\mathrm{Pr} \end{bmatrix}, \quad K = \sqrt{RM}, \tag{6}$$

and $\mathbf{X} = \mathbf{X}(\omega, \Theta)$ is a two-dimensional vector.

If the time-dependence of the vector in Eq. (5) is exponential, $\exp(\lambda\tau)$, then we have the generalized eigenvalue problem in the parameter λ:

$$A\mathbf{X} - \lambda B\mathbf{X} = 0. \tag{7}$$

For a symmetric matrix $A(T_0' = -1)$ and with sign-definite operators in elements of the matrices A and B ($\alpha = 0$), the criteria of orthogonality and normalization for the eigenvectors in problem (7) with homogeneous boundary conditions are defined as follows:

$$N_0 = (\mathbf{X}_1, B\mathbf{X}_2) = -\frac{1}{K}(\omega_1, D\omega_2) + \mathrm{Pr}\,K(\Theta_1, \Theta_2). \tag{8}$$

If the approximation μ to an eigenvalue of problem (7) has been found by any means whatever, then the eigenvalue can be found more exactly by the Wielandt method [4]. The gist of this method is to seek the smallest eigenvalue $\eta = \lambda - \mu$ of the shifted matrix $A - \mu B$, where λ denotes the exact eigenvalue of the original problem.

With all this in view, problem (7) of finding an eigenvalue more exactly becomes

$$(A - \mu B)\mathbf{X} - \eta B\mathbf{X} = 0, \tag{9}$$

and the corresponding Eq. (5) becomes

$$(A - \mu B)\mathbf{X} - B\frac{\partial}{\partial\tau}\mathbf{X} = 0. \tag{10}$$

2. For a numerical determination of the first few eigenvalues of problem (7) by iteration methods, it is worthwhile to consider the finite-differences approximation to the transient equation (5). If a step along the time coordinate is denoted by $\Delta\tau$ and a step along the space coordinate is denoted by h, then the finite-differences equation corresponding to (5) will be

$$-\frac{1}{\Delta\tau}(B\mathbf{X})_i^j + \varepsilon(A\mathbf{X})_i^j = -\frac{1}{\Delta\tau}(B\overline{\mathbf{X}})_i^{j-1} + (\varepsilon - 1)(A\overline{\mathbf{X}})_i^{j-1}$$
$$(j = 1, 2, 3, \ldots;\ i = 0, 1, 2, \ldots, N), \tag{11}$$

where a superscript and a subscript indicate a discrete variation of a given quantity along the time coordinate and along the space coordinate, respectively, N is the number of points on the interval, and the real parameter ε can vary within $0 \le \varepsilon \le 1$.

With the aid of Eq. (11) one can use the stabilization method [5] for determining the first eigenvector (regular solution ~exp ($\lambda\tau$)) which corresponds to the largest eigenvalue. Taking an arbitrary vector $\mathbf{X}^0$ as the start, we obtain a discrete in time sequence of values $\mathbf{X}^j$ (j = 1, 2, 3, ...) (analogous to the iteration sequence for the matrix of an arbitrary vector in the power method of solving the partial eigenvalues problem in [4]). If the process of determining $\mathbf{X}^j$ from Eq. (11) has stabilized, then the first eigenvalue is equal to the ratio of the respective components of vectors which represent the succeeding and the preceding approximation (the latter normalized in a definite manner): $\lambda_p = x_i^j/\bar{x}_i^{j-1}$ (the dash over the symbol denotes a normalized quantity). On the other hand, in this case $\mathbf{X}^j$, $\mathbf{X}^{j-1}$ satisfy Eq. (7). Inserting $(AX)_i^j$ and $(A\bar{X})_i^{j-1}$ from (7) into (11) and using operator B^{-1} on the left-hand side, we obtain the following relation between λ_p and λ:

$$\lambda_p = \frac{1+(1-\varepsilon)\,\lambda\Delta\tau}{1-\varepsilon\lambda\Delta\tau}. \tag{12}$$

Inasmuch as the elements of matrix A in (6) contain fourth-order differential operators, some terms of the finite-differences equation (11) will contain the factor h^4. As a result, in a digital computer designed for 7-decimal numbers (Minsk-22) some significant digits will be lost.

Therefore, for computer-technical reasons, it will be worthwhile to replace the fourth-order system (1) by an equivalent system of second-order equations. This is achieved by introducing a new function $\varphi(z, \tau)$ according to the equation

$$D\omega = \varphi. \tag{13}$$

With this substitution, then, (5), (7) and (9), (10) retain their form and only matrices A, B as well as vector $\mathbf{X}$ are now redefined as

$$A = \begin{bmatrix} D & -1 & 0 \\ 0 & D_3 & -K^2 \\ T_0' & 0 & -D_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\mathrm{Pr} \end{bmatrix}, \quad \mathbf{X} = \mathbf{X}(\omega, \varphi, \Theta), \tag{14}$$

while relation (8) for this case becomes

$$N_0 = -\int_0^1 \left(\frac{1}{K}\, \omega_1\varphi_2 - \mathrm{Pr}\, K\, \Theta_1\Theta_2 \right) dz. \tag{15}$$

If in Eq. (11), with A replaced by $A-\mu B$, the operators on the space coordinate in elements of the matrices A and B in (14) are replaced by symmetric three-point differences, then, after a few transformations, we obtain the following recurrent equation:

$$A_1\mathbf{X}_{i+1}^j + B_1\mathbf{X}_i^j + C_1\mathbf{X}_{i-1}^j = A_2\bar{\mathbf{X}}_{i+1}^{j-1} + B_2\bar{\mathbf{X}}_i^{j-1} + C_2\bar{\mathbf{X}}_{i-1}^{j-1} \tag{16}$$
$$(j=1,\ 2,\ 3,\ \ldots;\ i=1,\ 2,\ 3,\ N-1),$$

where A_1, B_1, C_1 are matrices of the third rank

$$A_1 = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{bmatrix}, \quad B_1 = \begin{bmatrix} \beta_{11} & \beta_{12} & 0 \\ 0 & \beta_{22} & \beta_{23} \\ \beta_{31} & 0 & \beta_{33} \end{bmatrix}, \quad C_1 = \begin{bmatrix} \gamma_{11} & 0 & 0 \\ 0 & \gamma_{22} & 0 \\ 0 & 0 & \gamma_{33} \end{bmatrix}, \tag{17}$$

and $\mathbf{X}^j = \mathbf{X}(\omega_i^j, \varphi_i^j, \Theta_i^j)$ is a three-dimensional vector.

The elements of matrix (17) are defined as follows:

$$\alpha_{11} = \gamma_{11} = \varepsilon;\ \alpha_{22},\ \gamma_{22} = \varepsilon\left(1 \pm \frac{\alpha h}{2\,\mathrm{Pr}}\right);\quad \alpha_{33},\ \gamma_{33} = -\varepsilon\left(1 \pm \frac{\alpha h}{2}\right);$$

$$\beta_{11} = -\varepsilon(2+h^2M^2);\ \beta_{12} = -\varepsilon h^2;\ \beta_{22} = -\frac{h^2}{\Delta\tau} - \varepsilon\left[2+(M^2+\mu)\,h^2\right]; \tag{18}$$

$$\beta_{23} = -\varepsilon h^2 M^2 \mathrm{R};\ \ \beta_{31} = \varepsilon h^2 (T_0')_i;\ \beta_{33} = \frac{\mathrm{Pr}\,h^2}{\Delta\tau} + \varepsilon\left[2+(M^2+\mathrm{Pr}\,\mu)\,h^2\right].$$

Matrices A_2, B_2, C_2 are analogous to matrices A_1, B_1, C_1, except that in Eqs. (18), which define their elements, ε must be replaced by $\varepsilon-1$.

In order to solve problem (16) with the respective boundary conditions we use the sweep method. We will seek the solution in the form of a recurrence relation

$$\mathbf{X}_i^j = P_i \mathbf{X}_{i+1}^j + \mathbf{Q}_i^j, \tag{19}$$

where the three-dimensional matrices P_i and vectors $\mathbf{Q}_i^j$ (i = 1, 2, 3, . . . , N−1) are to be determined in the first stage of the sweep (forward sweep).

The recurrence formulas for P_i, $\mathbf{Q}_i^j$ (i = 1, 2, 3, . . . , N−1) are

$$\begin{gathered} P_i = -(B_1 + C_1 P_{i-1})^{-1} A_1, \\ \mathbf{Q}_i^j = (B_1 + C_1 P_{i-1})^{-1} (\mathbf{R}_i^{j-1} - C_1 \mathbf{Q}_{i-1}^j), \\ \mathbf{R}_i^{j-1} = A_2 \overline{\mathbf{X}}_{i+1}^{j-1} + B_2 \overline{\mathbf{X}}_i^{j-1} + C_2 \overline{\mathbf{X}}_{i-1}^{j-1} . \end{gathered} \tag{20}$$

The initial value P_0 of the matrix and $\mathbf{Q}_0^j$ of the vector will be determined from the corresponding constraint on the left-hand boundary.

From the backward sweep we find $\mathbf{X}_N^j$ according to the recurrence formula (19) from the constraint on the right-hand side and from relation (19) for point i = N−1.

3. In order to determine the first three eigenvectors and the corresponding eigenvalues, we use the fully stabilizing step-by-step method [4]. The gist of this method, as applied to our scheme for solving Eqs. (16), is as follows.

Let $\mathbf{X}_1^0$, $\mathbf{X}_2^0$, $\mathbf{X}_3^0$ be three arbitrary linearly independent 3(N + 1)-dimensional vectors. We construct a sequence to the system using three mutually orthogonal vectors $\mathbf{X}_1^j$, $\mathbf{X}_2^j$, $\mathbf{X}_3^j$ (j = 1, 2, 3, . . .) which represent sequential in time approximations to the initial vectors $\mathbf{X}_1^0$, $\mathbf{X}_2^0$, $\mathbf{X}_3^0$, according to the adopted method of solving the finite-differences equations (16) with the corresponding boundary conditions.

The process of finding the said frequency will be denoted by operator A so that

$$\mathbf{X}_i^j = A \overline{\mathbf{X}}_i^{j-1} \quad (i = 1,\ 2,\ 3;\ j = 1,\ 2,\ 3,\ \ldots). \tag{21}$$

In order to avoid an increase in the number of components, it is advisable in the calculation by any one method to assign numbers to the vectors obtained in each step. Orthogonalization and normalization of the vectors is in our case effected in the generalized sense according to relation (15), with the integration replaced by a summation over subdivision points i = 0, 1, 2, . . . , N of the interval.

On the basis of the theorem in ([4], p. 384), the sequence to the system of vectors $\mathbf{X}_1^j$, $\mathbf{X}_2^j$, $\mathbf{X}_3^j$ has the limits $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ and these limit vectors lie in an invariant subspace which extends over a system of three eigenvectors corresponding to the first three eigenvalues λ_{p1}, λ_{p2}, λ_{p3} or over a system of one eigenvector and two root vectors corresponding to equal eigenvalues.

Thus the problem has been reduced to the complete eigenvalue problem in a given three-dimensional subspace.

As the basis of this system we will choose vectors $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$. Then

$$A \mathbf{X}_n = \sum_{m=1}^{3} \alpha_{mn} \mathbf{X}_m \quad (n = 1,\ 2,\ 3), \tag{22}$$

so that the matrix

$$L = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \tag{23}$$

is a matrix of the induced operator in the chosen subspace. By virtue of the orthogonality of vectors $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, the elements of matrix L are related as follows:

$$\alpha_{mn} = (A \mathbf{X}_n,\ \mathbf{X}_m). \tag{24}$$

The sought eigenvalues of problem (16) coincide with the eigenvalues of matrix L, which have been denoted by λ_{pn} (n = 1, 2, 3). Let the corresponding eigenvectors of matrix L be $\mathbf{a}_n = \mathbf{a}_n(a_{n1},\ a_{n2},\ a_{n3})$. In the basis $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, then, the coordinates of the eigenvectors of problem (16) $\mathbf{U}_1$, $\mathbf{U}_2$, $\mathbf{U}_3$, which correspond to the sought eigenvalues, are equal to the coordinates of the eigenvectors (or one eigenvector and two root vectors) of matrix L, namely:

$$\mathbf{U}_n = \sum_{m=1}^{3} a_{nm} \mathbf{X}_m \quad (n = 1,\ 2,\ 3). \tag{25}$$

TABLE 1. Parameter R in the Rayleigh Problem, as a Function of the Wave Number M on the Neutral Line in the Critical Range, for Various Step Sizes h in the Finite-Differences Schedule

M	h						Exact solution [1]
	0,02	0,01	0,008(3)	0,00625	0,00(5)	0,005	
3,00	1705,9	1709,9	1710,3	1710,7	1710,8	1710,8	1711,2
3,13	1702,5	1706,5	1706,9	1707,3	1707,5	1707,4	1707,8
3,26	1707,6	1711,4	1712,1	1712,5	1712,6	1712,5	1713,0

Indeed, if in the relation

$$A\mathbf{U}_n = \lambda_{pn}\mathbf{U}_n \quad (n=1,\ 2,\ 3) \tag{26}$$

we express the eigenvectors $\mathbf{U}_n$ ($n = 1, 2, 3$) according to formula (25), then with the aid of relation (22) we obtain the equality

$$\sum_{l=1}^{3}\left(\sum_{m=1}^{3}\alpha_{lm}a_{nm}-\lambda_{pn}a_{nl}\right)\mathbf{X}_l=0 \quad (n=1,\ 2,\ 3),$$

and from here, by virtue of vectors $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ being linearly independent,

$$\sum_{m=1}^{3}\alpha_{lm}a_{nm}=\lambda_{pn}a_{nl} \quad (l,\ n=1,\ 2,\ 3),$$

or

$$L\,\mathbf{a}_n=\lambda_{pn}\,\mathbf{a}_n \quad (n=1,\ 2,\ 3),$$

indicating that $\mathbf{a}_n$ is, indeed, the eigenvector of matrix L which corresponds to the eigenvalue λ_{pn} ($n = 1$, 2, 3).

The accuracy with which the eigenvectors are determined by this method can be readily estimated on the basis of the variance between a succeeding and a preceding iteration. Denoting the vector variance by $\vec{\Delta}_n$ and its norm by δ_n, we obtain the relation

$$\vec{\Delta}_n = A\mathbf{U}_n^j-\lambda_{pn}\,\overline{\mathbf{U}}_n^j,\quad \delta_n=\sqrt{(\vec{\Delta}_n,\ \vec{\Delta}_n)} \quad (n=1,\ 2,\ 3), \tag{27}$$

where the scalar product, as everywhere here, is taken in accordance with expression (15).

The real eigenvalue λ_n of problem (7), which corresponds to the eigenvalue λ_{pn} ($n = 1, 2, 3$) of the finite-differences problem, will be determined with the aid of (12).

We note that the speed at which the vectors stabilize by this method depends on the ratio λ_4/λ_3.

4. a) For the Rayleigh problem with zero boundary value of velocity, of its first derivative, and of temperature we determine from (13) the initial values P_0, $\mathbf{Q}_0^j$ for the forward sweep and $\mathbf{X}_N^j$ for the backward sweep:

$$P_0=\begin{bmatrix}0 & 0 & 0\\ 2/h^2 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad \begin{aligned}&\mathbf{Q}_0^j=\mathbf{Q}_0^j(0,\ 0,\ 0),\ \mathbf{X}_N^j=\mathbf{X}_N^j(0,\ \varphi_N,\ 0),\\ &\varphi_N=q_{N-1}(1)/\left[\frac{h^2}{2}-p_{N-1}(1,\ 2)\right].\end{aligned} \tag{28}$$

For boundary conditions of the third kind [2]

$$-\frac{\partial\Theta}{\partial z}=\mp\mathrm{Bi}\,\Theta \quad \text{при } z=0,\ 1$$

we determine the corresponding values P_0, $\mathbf{Q}_0^j$, and $\mathbf{X}_N^j$ as follows:

$$P_0=\begin{bmatrix}0 & 0 & 0\\ p_0(2,\ 1) & 0 & 0\\ 0 & 0 & p_0(3,\ 3)\end{bmatrix}\quad \begin{aligned}&p_0(2,\ 1)=2/h^2,\ p_0(3,\ 3)=1/(1+h\,\mathrm{Bi}),\\ &\mathbf{Q}_0^j=\mathbf{Q}_0^j(0,\ 0,\ 0),\ \mathbf{X}_N^j=\mathbf{X}_N^j(0,\ \varphi_N,\ \Theta_N),\end{aligned} \tag{29}$$

where φ_N and Θ_N are found by solving the algebraic system

$$\left[p_N(1,\ 2)-\frac{h^2}{2}\right]\varphi_N+p_{N-1}(1,\ 3)\,\Theta_N=-q_{N-1}(1),$$
$$p_{N-1}(3,\ 2)\,\varphi_N+\left[p_{N-1}(3,\ 3)-(1+h\,\mathrm{Bi})\right]\Theta_N=-q_{N-1}(3).$$

TABLE 2. First Eigenvalue λ_1 for the Critical R^* and M^* Numbers at Various Values of the Biot Number

Bi	0	0,1	0,25	0,5	1	5	10	100
M	1,25	1,40	1,75	2,0	2,25	2,75	3,01	3,13
R	740	840	920	1000	1100	1400	1550	1707
λ_1	—0,057	—0,005	0,030	0,042	0,024	0,025	0,352	0,245

TABLE 3. Parameter R as a Function of the Wave Number M on the Neutral Line, at the Value of the Injection Parameter α = 3

M	2,9	3,0	3,1	3,2	3,35	3,5	3,65
R	2612	2584	2565	2554	2550	2561	2587

The problems in this article were solved for the parameters $\alpha = 0$ and Pr = 0.7 by the implicit scheme $\varepsilon = 1$.

The computer values of parameter R in the Rayleigh problem are shown in Table 1 for three values of the wave number M within the critical range, with various step h sizes in the finite-differences scheme. The first eigenvalue λ_1 was reduced toward zero ($\sim 10^{-3}$), the norm of the variance for the first eigenvector was set at $\delta_1 = 10^{-6}$, while for the second and third eigenvector we obtained $\delta_{1,2} \sim 10^{-4}$–$10^{-5}$. The initial value of vectors $\mathbf{X}_n^0$ (n = 1, 2, 3) for this calculation were chosen arbitrarily, and then for the other parameters R and M we chose as the initial values the eigenvectors found for the preceding values of these parameters. This ensured the necessary accuracy after 1-2 iterations, while the step along the time coordinate could be chosen of any length between 0.1 and 4. In the last column of Table 1 are shown the results of an exact solution [1].

The values of parameter R in the problem with boundary conditions of the third kind were taken from [2], as were also the corresponding values of the wave number M on the line of minimum Biot number, and the first three corresponding eigenvalues of the problem were determined for these parameter values and with h = 0.00625. Only the first eigenvalue is shown in Table 2. As can be seen here, $\lambda_1 \sim 10^{-1}$–10^{-2}.

As was to be expected, the induced matrix L in these problems is a diagonal one, since the vectors of sequence (21) have been reduced to orthogonality in the generalized sense (15), since the original matrix A in (6) is symmetric, and since the differential operators in elements of the matrices A, B are sign-definite.

b) The problem of hydrodynamic stability with ejection [3] was solved for the boundary conditions (28) and the parameters $\alpha = 3$, Pr = 1 with $\varepsilon = 1$ and h = 0.01. The values of M and R on the neutral line are shown in Table 3. According to the data in Table 3, the minimum value is $R^* = 2550$ and corresponds to $M^* = 3.31$. Inasmuch as the characteristic dimension in [3] is one half of the characteristic length stipulated here, the parameter values referred to one half of our dimension will be $\alpha = 1.5$, $R^* = 160$, and $M^* = 1.66$, which agrees with the data in [3].

NOTATION

Pr is the Prandtl number;
$\alpha = w_0 H/a$ is the Peclet number;
Bi is the Biot number;
$\Delta T = T_1 - T_2$ is the temperature difference between the lower and upper plane;
a is the thermal diffusivity;
ν is the kinematic viscosity;
$\beta = -(1/\rho)(\partial\rho/\partial T)_p$ is the thermal volume expansivity;
$R = g\beta H^2 \Delta T/a\nu$ is the Rayleigh number;
g is the acceleration of free fall;
H is the height of the liquid layer;
$q_N(n)$ (n = 1, 2, 3) is the component of vector $\mathbf{Q}_N$;
$p_N(n, m)$ are the elements of the matrix P_N;
n, m are the number of the row and the column, respectively (n, m = 1, 2, 3);
$D = \partial/\partial z - M^2$;
$D_1 = D_3 D$;
$D_2 = \alpha(\partial/\partial z) + D$;
$D_3 = (\alpha/Pr)(\partial/\partial z) + D$.

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